

1. Find the first and second derivatives of the following functions:

a. $f(x) = \frac{5\sin(x)}{3} - x$
 $\frac{d}{dx}(\cos(x)) = -\sin(x)$
 $\frac{d}{dx}(\sin(x)) = \cos(x)$

$$f'(x) = \frac{5\cos(x)}{3} - 1$$

$$f''(x) = \frac{-5\sin(x)}{3}$$

b. $y = \frac{x}{2} - \frac{\sin(2x)}{4}$

$$\frac{d}{dx}(\sin(2x)) = 2\cos(2x) \dots \text{differentiating using chain rule where } u = 2x$$

$$y'(x) = \frac{1}{2} - \frac{\cos(2x)}{2}$$

$$\frac{d}{dx}(\cos(2x)) = -2\sin(2x) \dots \text{differentiating using chain rule where } u = 2x$$

$$y''(x) = \sin(2x)$$

2. Find the derivative:

a. $y = \sec^2(\pi x)$

$$\sec^2(\pi x) = (\sec(\pi x))^2$$

$$\frac{d}{dx}(\sec(\pi x)) = \sec(\pi x)\tan(\pi x) * \frac{d}{dx}(\pi x)$$

$$= \pi \sec(\pi x) \tan(\pi x)$$

Applying Chain Rule: $u = \sec(\pi x)$

$$\frac{d}{dx}(\sec^2(\pi x)) = 2(u)(u')$$

$$= 2(\sec(\pi x)) (\pi \sec(\pi x) \tan(\pi x))$$

$$= 2\pi \sec^2(\pi x) \tan(\pi x)$$

b. $y = \sin^2\left(\frac{x}{2}\right)$

$$\sin^2\left(\frac{x}{2}\right) = \left(\sin\left(\frac{x}{2}\right)\right)^2$$

Applying Chain Rule: $u = \sin\left(\frac{x}{2}\right)$

$$\frac{du}{dx} = \frac{1}{2} \cos\left(\frac{x}{2}\right)$$

$$\frac{d}{dx}(\sin^2\left(\frac{x}{2}\right)) = 2(u) \cdot u'$$

$$= \sin\left(\frac{x}{2}\right) \left(\cos\left(\frac{x}{2}\right)\right)$$

$$= \frac{\sin(x)}{2}$$

3. $f(x) = \frac{2}{\sqrt{x+1}}$

a. Find $f'(x)$

$$\frac{2}{\sqrt{x+1}} = 2(x+1)^{-\frac{1}{2}}$$

Applying Chain Rule: $u = x+1$

$$\frac{d}{dx} (2(x+1)^{-\frac{1}{2}}) = -1(x+1)^{-\frac{3}{2}}$$

$$= \frac{-1}{(x+1)^{\frac{3}{2}}}$$

b. Find the point on f where the slope of the tangent is $-\frac{1}{8}$.

$$\frac{-1}{(x+1)^{\frac{3}{2}}} = -\frac{1}{8}$$

$$(x+1)^{\frac{3}{2}} = 8$$

$$(x+1)^3 = 64$$

$$x+1 = 4$$

$$x = 3$$

4. $f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 1$

a. Find the derivative

$$F'(x) = x^2 - 3x$$

b. Find all points $(x, f(x))$ on the graph of $f(x)$ with tangent lines parallel to the line $8x - 2y = 1$

$$8x - 2y = 1$$

$$y = 4x - \frac{1}{2}$$

$$m = 4$$

$$4 = x^2 - 3x$$

Solving the quadratic equation, $x = 4$, $x = -1$

$$f(4) = \frac{4^3}{3} - \frac{3(4)^2}{2} + 1 = \frac{-5}{3}$$

$$f(-1) = \frac{(-1)^3}{3} - \frac{(-1)^2}{2} + 1 = \frac{-5}{6}$$

$$= (4, \frac{-5}{3}) \text{ and } (-1, \frac{-5}{6})$$

5. $f(x) = \frac{27}{x^2+2}$

a. Find its derivative.

Applying Quotient Rule:

$$\begin{aligned}\frac{d}{dx}\left(\frac{27}{x^2+2}\right) &= \frac{-(2x)*27}{(x^2+2)^2} \\ &= \frac{-54x}{(x^2+2)^2}\end{aligned}$$

b. Find the equation of the tangent line at $x = 1$

$$f(1) = 9$$

$$m = f'(1) = \frac{-54}{(1+2)^2} = -6$$

$$m - (9) = -6(x - 1)$$

$$y = -6x + 15$$

6. $f(x) = 1 - \cos(2x) + \cos^2(x)$

a. Find the derivative.

$$f'(x) = (-2\sin(2x)) - 2\sin(x)\cos(x)$$

$$= 2\sin(2x) - \sin(2x)$$

$$= \sin(2x)$$

b. Determine the values of x in the interval $[0, 2\pi]$ at which the graph of $f(x)$ has a horizontal tangent lines.

$$\sin(2x) = 0$$

$$2x = 0; x = 0$$

$$2x = \pi; x = \frac{\pi}{2}$$

$$2x = 2\pi; x = \pi$$

$$2x = 3\pi; x = \frac{3\pi}{2}$$

$$2x = 4\pi; x = 2\pi$$

$$x = 0, x = \frac{\pi}{2}, x = \pi, x = \frac{3\pi}{2}, x = 2\pi$$

7. **Given:** $6x^2y - \pi\cos(y) = 7\pi$

a. Find $\frac{dy}{dx}$.

$$\frac{d}{dx}(6x^2y) - \frac{d}{dx}(\pi\cos(y)) = \frac{d}{dx}(7\pi)$$

$$12xy + 6x^2 \frac{dy}{dx} + \pi\sin(y) \frac{dy}{dx} = 0$$

$$6x^2 \frac{dy}{dx} + \pi\sin(y) \frac{dy}{dx} = -12xy$$

$$\frac{dy}{dx} = \frac{-12xy}{6x^2 + \pi\sin(y)}$$

b. Find the equation of the normal line at the point $(1, \pi)$

$$M_{\text{tangent line}} = \frac{-12\pi}{6 + \pi\sin(\pi)} = -2\pi$$

$$M_{\text{normal line}} = \frac{1}{2\pi}$$

$$y - (\pi) = \frac{1}{2\pi}(x - 1)$$

$$y = \frac{1}{2\pi}(x+1)$$

8. A 17-ft ladder is leaning against the side of a house. The top of the ladder is sliding down the house at rate of 5 ft/sec.

a. Determine how fast the bottom of the ladder is sliding away when the top of the ladder is 8 feet from the ground.

$$x^2 + y^2 = c^2$$

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(c^2)$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2c \frac{dc}{dt}, \text{ but } \frac{dc}{dt} = 0$$

$$(8)(-5) + 15\left(\frac{dy}{dt}\right) = 0$$

$$\frac{dy}{dt} = \frac{8}{3} \text{ ft/sec}$$

- b. **At what rate is the angle θ between the ladder and the ground is changing then.**

$$\frac{y}{x} = \tan\theta$$

$$y = x \tan\theta$$

$$\frac{dy}{dt} = \tan\theta \frac{dx}{dt} + x \sec^2\theta \frac{d\theta}{dt}$$

$$-5 = \left(\frac{8}{15}\right)\left(\frac{8}{3}\right) + (15)\left(\frac{17}{15}\right)^2 \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{-1}{3} \text{ rad/s}$$

9. **A cube's surface area increases at a rate of $72 \text{ in}^2 / \text{sec}$. At what rate is the cube's Volume changing when the side's length is 3 inches?**

$$S = 6a^2$$

$$\frac{dS}{dt} = 12a \frac{da}{dt}$$

$$\frac{dS}{dt} = 72 \text{ in}^2 / \text{sec}$$

$$72 = 12a \frac{da}{dt}; a = 3$$

$$\frac{da}{dt} = 2 \text{ in/sec}$$

$$V = a^3$$

$$\frac{dV}{dt} = 3a^2 * \frac{da}{dt}$$

$$\frac{dV}{dt} = 3 * 9 * 2$$

$$=54 \text{ in}^3/\text{sec}$$

- 10) A water tank has the shape of an inverted right circular cone. The container has a radius of 2 m and a height of 4m. If water is being pumped into the tank at a rate of 2 m³/sec. Find the rate at which water level is rising when the water is 3 m deep.

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$V = \frac{1}{12}\pi h^3$$

$$\frac{dV}{dt} = \frac{dV}{dh} * \frac{dh}{dt}$$

$$2 = \frac{d}{dh}\left(\frac{1}{12}\pi h^3\right) \frac{dh}{dt}$$

$$8 = \pi h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{8}{\pi h^2} \text{ where } h = 3$$

$$\frac{dh}{dt} = \frac{8}{9\pi} \text{ m/min}$$